Distribution of the Masses of Protostars in Globular Clusters

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Larson identifies the globular star clusters as the original products of the star formation process. In *The Neglected Facts of Science*,[1] he gives an account of this process: "…consider successively larger spherical aggregates of dispersed matter: … the particles of this matter … are moving outward away from each other by reason of the progression of the natural reference system. Coincidentally they are moving inward toward each other gravitationally, and also inward toward the center of the aggregate under the gravitational influence of the aggregate as a whole. In the central regions of this aggregate, the net motion is outward, but the gravitational effect on the outer particles increases with the radius of the sphere, and at some very large distance, the inward and outward motions reach equality. Beyond this distance, the net motion is inward…

"While the end result of this process is an equilibrium, not a condensation, the action does not stop at this point. … there is a continuous inflow of matter from the cosmic sector of the universe. This matter is dispersed throughout all of the space of the material sector, and the mass contained within the equilibrium system is therefore slowly increased. This strengthens the gravitational forces and initiates a contraction of the aggregate. … Meanwhile … the original aggregate separates into a large group of sub-aggregates. Eventually, the sub-aggregates become stars, and the aggregate as a whole becomes a globular cluster."

We will now attempt to theoretically arrive at the protostar mass distribution in the inchoate globular cluster by applying the principles of the Reciprocal System.

1 Force of the Space-Time Progression

In the context of the familiar three-dimensional stationary spatial reference system the outward motion of the space-time progression (STP) originates at every point of space and is independent of distance. It depends only on the number of mass units (that is, the number of space-time locations occupied by the individual mass units) of a material aggregate. Let *Y* be the force of STP on a unit mass, in dynes/gram. Consider a large mass *M*, in the vicinity of the unit mass but not overlapping the latter. We may write the net force exerted by the STP and the gravitation of the mass *M* on the unit mass as

$$a = Y - \left(\frac{G\,M}{x^2}\right) \quad dyne/g \text{ or } cm/s^2 \tag{1}$$

where

x = the distance between the unit mass and the mass M, and
G = the gravitational constant.

Since at the inner gravitational limit R_{il} of the mass *M*, $a = 0$, we have

1 Larson, Dewey B., *The Neglected Facts of Science*, North Pacific Pub., Portland, OR, 1982, p. 115.

$$Y = \frac{G\,M}{R_{il}^2} \tag{2}$$

From Larson[2] we have, in natural units,

$$\frac{M}{R_{il}^2} = 9\left(156.444\right)^4$$

Therefore, in conventional units,

$$\begin{aligned} Y &= \frac{(6.673\times10^{-8})(9\times156.44^4)(1.65979\times10^{-24})}{(4.558816\times10^{-6})^2} \\ &= 2.87311647\times10^{-11} \quad \mathit{dyne/g} \end{aligned} \tag{3}$$

2 Potential Energy

The formation of the globular cluster of stars from a cloud of dust and gas is principally determined by the potential energy (virial) of the cloud of dispersed matter. We will assume a spherical mass of radius R and of density ρ. The postulates of the *Reciprocal System* do not lead us to assume anything other than that this initial density is uniform. They also suggest that the role of factors other than gravitation and STP, like rotation, for example, in determining the initial course of evolution of these large dispersed aggregates is insignificant. As such, the net (outward plus inward) force per unit mass at a radius of x in the sphere is

$$a = Y - \left(\frac{G\,M}{R^3}\right)x = Y - \left(\frac{4}{3}\pi\,G\,\rho\,x\right) \tag{4}$$

The potential energy dP of a spherical shell of matter of radius x and thickness dx is then given by

$$dP = \left(\rho\cdot 4\pi\,x^2\cdot dx\right)\left(Y - \frac{4}{3}\pi\,G\,\rho\,x\right)x \tag{5}$$

Integrating between the limits $x = 0$ and $x = R$, the total potential energy of the spherical mass is given by

$$P = \pi\,Y\,\rho\,R^4 - \frac{16}{15}\pi^2 G\,\rho^2 R^5 \tag{6}$$

The spherical aggregate either expands, remains in equilibrium, or contracts, depending on whether P is positive, zero or negative respectively. Solving for the radius for which P becomes zero, therefore, gives us the critical radius R_{cr} of the mass M, for it to be in equilibrium. Thus, from Equation (6),

$$\rho\cdot R_{cr} = \frac{15}{16}\frac{Y}{\pi\,G} \tag{7}$$

2 Larson, *The Structure of the Physical Universe*, (North Pacific Pub., Portland, OR, 1959), p 166.

Since $R = (3M / 4\pi\,\rho)^{1/3}$ we have the following useful relations:

$$M_{cr} = \frac{8.8848 \times 10^{-12}}{\rho_{cr}^2} \quad grams \tag{8}$$

and

$$R_{cr} = 2.0326 \sqrt{\frac{M_{cr}}{M_\odot}} \quad light\text{-}years \tag{9}$$

where $M_\odot$ is the solar mass and ρ is in g/cm^3.

For instance, at a density of 10^{-25} g/cm^3, the critical radius of the cloud for equilibrium turns out to be nearly 416 parsecs. The total mass of the cloud is $4.5 \times 10^5\ M_\odot$.

Further, in such an uniformly dispersed spherical mass there is a net outward motion up to a radius of R_i and a net inward motion beyond it. From Equations (4) and (7) we find

$$R_i = \frac{4}{5} R_{cr} \tag{10}$$

3 Fragmentation

Fragmentation to form protostars can get initiated only within the portion of the prestellar cloud between the radii and R_i and R_{cr} since the net motion in this zone is inward. In order to arrive at the mass spectrum of the protostars that form in this portion, it is convenient to resort to the concept of potential, or the potential energy per unit mass, σ. From Equation (6) we obtain the net potential (STP and gravitation) of a spherical mass *m* as

$$\sigma_x = B \cdot m^{1/3} - A \cdot m^{2/3} \quad cm^2/s^2 \tag{11}$$

where *A* and *B* are constants dependent on ρ, the density. Further, from Equation (5) we note that the potential at any distance *x* from the center of the spherical mass is

$$\sigma_x = \left(Y - \frac{4}{3} \pi G \rho\, x \right) x \quad cm^2/s^2 \tag{12}$$

σ_x represents the net motion at the location *x*. This is reckoned from the Reference Point[3] posited as situated at the center of the spherical cloud. Remembering that the motion is intrinsically scalar in nature, we recognize that it is independent of the Reference Point. As such, it can equivalently be reckoned from the Reference Point taken as situated at location *x* (or, in fact, from a multitude of Reference Points located over the spherical surface of radius *x*). Thus, it can manifest as the potential of an individual mass *m*, or of a spectrum of masses, <u>at location *x*</u>. Consequently, at a given radius *x* in the prestellar cloud, fragmentation occurs, resulting in protostars with an upper limit for mass given by (from Equations (11) and (12))

$$\sigma_x = B \cdot m_{max}^{1/3} - A \cdot m_{max}^{2/3} \tag{13}$$

3 Larson, *The Neglected Facts of Science*, *op. cit.*, p. 7.

It is worthwhile to note that no separate theory is necessary to explain the phenomenon of fragmentation. In the Reciprocal System, it follows directly from one of the principal characteristics of scalar motion which is the fundamental component of the physical universe, namely, its independence of the Reference Point, together with the fact that the probability of the multiple Reference Points dispersed throughout the volume is greater than that of a single Reference Point situated at the center of the gravitating spherical aggregate.

4 Mass Spectrum of the Protostars

At a radius x in the original cloud, the specific potential energy, σ, can assume a value such that

$$0 \leq \sigma \leq \sigma_x \tag{14}$$

with the consequent fragment masses ranging from m_1, the smallest to m_2, the largest. In order to find the relative distribution of the stellar masses we have to know the probability of formation of a fragment with mass m, or equivalently, the probability of occupancy of the corresponding potential level, σ.

So as to evaluate this probability, we draw on an analogy with the Maxwell-Boltzmann distribution applied to the case of indistinguishable particles occupying continuously variable energy level ε and in thermal equilibrium with an environment at temperature T. The probability of relative occupancy of an energy level range between ε and $\varepsilon + d\varepsilon$ is given by

$$p(\varepsilon) = J \cdot e^{\left(\frac{-\varepsilon}{kT}\right)} \cdot d\left(\frac{\varepsilon}{kT}\right)$$

where k is the Boltzmann's constant and J a normalization constant.

In the present case, individual mass units take the place of particles; $\sqrt{\sigma/\sigma_x}$ takes the place of ε/kT since $\sqrt{\sigma}$ is the measure of the scalar motion (in speed units) and $\sqrt{\sigma_x}$ represents the motion level of the environment. Hence, the probability of occupancy of the motion level represented by σ is given by

$$\begin{aligned} p(\sigma) &= J_1 \cdot e^{-\sqrt{\sigma/\sigma_x}} \cdot d\left(\sqrt{\sigma/\sigma_x}\right) \\ &= J_2 \frac{e^{-\sqrt{\sigma/\sigma_x}}}{\sqrt{\sigma/\sigma_x}} d\sigma \end{aligned} \tag{15}$$

where J_2 is the normalization constant to be evaluated later.

Let $N(>m)$, or simply N, represent the total number of fragments (protostars) with masses greater than m, in the entire cloud (with $R_i \leq x \leq R_2$, R_2 being the outer radius which may be equal to R_{cr}, or less than R_{cr} if there has been some amount of cloud contraction). Let $\sigma_x N$ be the number of fragments with masses greater than m at a given radius x and $\delta_\sigma \delta_x N$, or simply $\delta\delta N$, the number of fragments at x and at the particular mass m.

Now the ratios of the mass units in the m-sized fragments at location x to the total number of mass units available at the same location, that is, the relative occupancy of the energy level pertaining to the mass m is

$$\frac{m \cdot \delta\delta N}{\rho \cdot 4\pi x^2 \cdot dx} \tag{16}$$

Equating this to Equation (15) we obtain

$$\delta\delta N = J\frac{x^2}{m\sqrt{\sigma\sigma_x}}e^{-\sqrt{\sigma/\sigma_x}}d\sigma dx \quad (17)$$

where $J = 4\pi\rho J_2$.

5 The Smallest and the Largest Stellar Masses

What decides whether a collapsing fragment will eventually become a star or not is the question of the initiation of thermal destruction. If the starting potential energy is not sufficient to raise the central temperature to the thermal destructive limit of the heaviest element present, the fragment fails to become a star. Probably the minimum mass required for this is on the order of 0.001 $M_\odot$; but its actual calculation depends on the knowledge of the destructive temperatures of the elements.

Even though the mass of the smallest fragment that can form in the cloud with density ρ is given by Equation (8)

$$m_{min} = \frac{8.8848\times10^{-12}}{\rho^2} \quad (8a)$$

the smallest mass m_1, that can actually survive is somewhat greater for the following reason. At the very early stage when fragmentation is just setting in, because of the propinquity, mutual gravitational capture takes place as a rule rather than an exception. Consider a small fragment (with mass *m*) surrounded by larger ones. If it falls within their gravitational limit it gets cannibalized by the latter. However, the fragment-*m* itself cannibalizes all of the fragments smaller than itself that come within the ambit of its own gravitational limit. Now the greatest mass loss a fragment can incur, by being cannibalized by its bigger brothers, cannot be greater than its own mass, *m*. Under these circumstances, a fragment can just survive when its mass loss (that is, *m*) equals the mass gain from its own capture of its circumambient smaller brothers. Consequently, there is a lower limit for the mass *m*, for survival to be possible. Let m_1 be this lower limit: we evaluate it as follows.

The gravitational limit of the mass m_1, is given by $K\sqrt{m_1}$, where $K = 2.27 \times 9.46053\times10^{17}$. Hence, the volume of the space surrounding the mass in which its gravitational capture is effective is

$$\frac{4}{3}\pi K^3 m_1^{3/2}$$

Since the density of the cloud ρ is uniform, the total mass contained in this volume is

$$\frac{4}{3}\frac{\pi K^3\rho}{M_\odot}m_1^{3/2} \quad \textit{(in solar units).}$$

But this includes the mass m_1 of the fragment as well. As such, the net mass contained in the gravitational volume of the fragment other than its own mass is

$$m_{net} = \left(\frac{4\pi K^3\rho m^{3/2}}{3M_\odot}\right) - m_1 \quad (18)$$

Of this, a certain fraction manifests as fragments with masses less than m_1, and the remaining as fragments with masses greater than or equal to m_1. Since only those fragments with masses less than m_1 are captured by fragment-m_1, we will compute their total mass now. From Equation (17), we can write that at any radius x in the cloud, the ratio of the mass contained in the range $\sigma_{min} \le \sigma \le \sigma_1$ (where σ_{min} is determined by m_{min} of Equation (8a), and σ_1 by m_1 through Equation (11)) to the mass contained in the range $\sigma_{min} \le \sigma \le \sigma_x$ as

$$Q = \frac{\int_{\sigma_{min}}^{\sigma_1} e^{-\sqrt{\sigma/\sigma_x}} \, d\sqrt{\sigma/\sigma_x}}{\int_{\sigma_{min}}^{\sigma_x} e^{-\sqrt{\sigma/\sigma_x}} \, d\sqrt{\sigma/\sigma_x}} = \frac{\left[1 - e^{\sqrt{\sigma_{min}/\sigma_x} - \sqrt{\sigma_1/\sigma_x}}\right]}{\left[1 - e^{\sqrt{\sigma_{min}/\sigma_x} - 1}\right]} \tag{19}$$

Multiplying this ratio by the net mass (Equation (18)) contained in the gravitational volume gives us the mass available, in the form of smaller fragments, for capture by m_1. Therefore, under the condition of marginal survival,

$$m_1 = \left[\frac{4\pi K^3 \rho}{3 M_\odot} m_1^{3/2} - m_1\right] \cdot Q$$

Or,

$$m_1 = \left[\frac{\left(1 + \frac{1}{Q}\right) \cdot 3 M_\odot}{4\pi K^3 \rho}\right]^2 \tag{20}$$

with Q substituted from Equation (19).

As an example, for an aggregate mass of $10^6\ M_\odot$, the values of the effective smallest mass m_1 calculated for the cloud densities 10^{-21}, 10^{-20} and 10^{-19} g/cm^3 turn out to be respectively 3.16, 0.18 and 0.01 $M_\odot$.

Even though the theoretical maximum mass limit is given by Equation (13), there is a statistical limit, m_2, that is much smaller than m_{max}. In view of the fact that the number of stars in any mass range can never be fractional or less than unity, m_2 is effectively given by

$$N(>m_2) = 1 \tag{21}$$

However, this value of m_2 may not also be of much significance, for, in all probability, this mass further sub-divides into smaller fragments soon afterwards.

6 The Initial Mass Function

Finally, from Equation (17), by integrating, we obtain the total number of stars with individual masses greater than m as

$$N(>m) = J \int_{R_1}^{R_2} \int_{\sigma_0}^{\sigma_x} \frac{x^2 \cdot e^{-\sqrt{\sigma/\sigma_x}}}{m \cdot \sqrt{\sigma/\sigma_x}} \, d\sigma \, dx \tag{22}$$

where σ_0 is set equal to σ or σ_1, depending on whichever is less. The lower limit R_1 is determined by m, through Equations (11) and (12).

Further, the mass $M(>m)$ of the stellar aggregate with individual masses greater than m is given by

$$M(>m) = J \int_{R_1}^{R_2} \int_{\sigma_0}^{\sigma_x} \frac{x^2 \cdot e^{-\sqrt{\sigma/\sigma_x}}}{\sqrt{\sigma \cdot \sigma_x}} \, d\sigma \, dx \tag{23}$$

It can be seen that the constant J can now be evaluated from the equation

$$M(>m_1) = M_t \tag{24}$$

where M_t = the total mass of the aggregate of stars = $\frac{4}{3}\pi\rho\left(R_2^3 - R_i^3\right)$.

The values of $N(>m)$ vs. m, obtained from Equation (22), are shown plotted in Figure 1 on log-log basis, for three different cloud densities. The plots indicate a unique lower mass limit m_1, for each. Further, it is apparent that the relation between N and m is of the type

$$\left.\begin{array}{l} N(>m) = constant \cdot m^{1-\beta} \\ \dfrac{dN}{dM} \alpha\, m^{-\beta} \end{array}\right\}, \; m \geq m_1 \tag{25}$$

The second of the above is referred to as the initial mass function. By linear regression, the value of β is found to be 2.34. Its value derived from astronomically observed data on star clusters is reported to be 2.35. However, it is necessary to remember that the latter derivation is based on certain *assumptions* chief of which are that stars generate their energy by the so-called hydrogen fusion process, that they evolve off the main sequence when 10 percent of their hydrogen gets converted to helium and that the rate of star formation has been unchanged for the past 10^9 years.

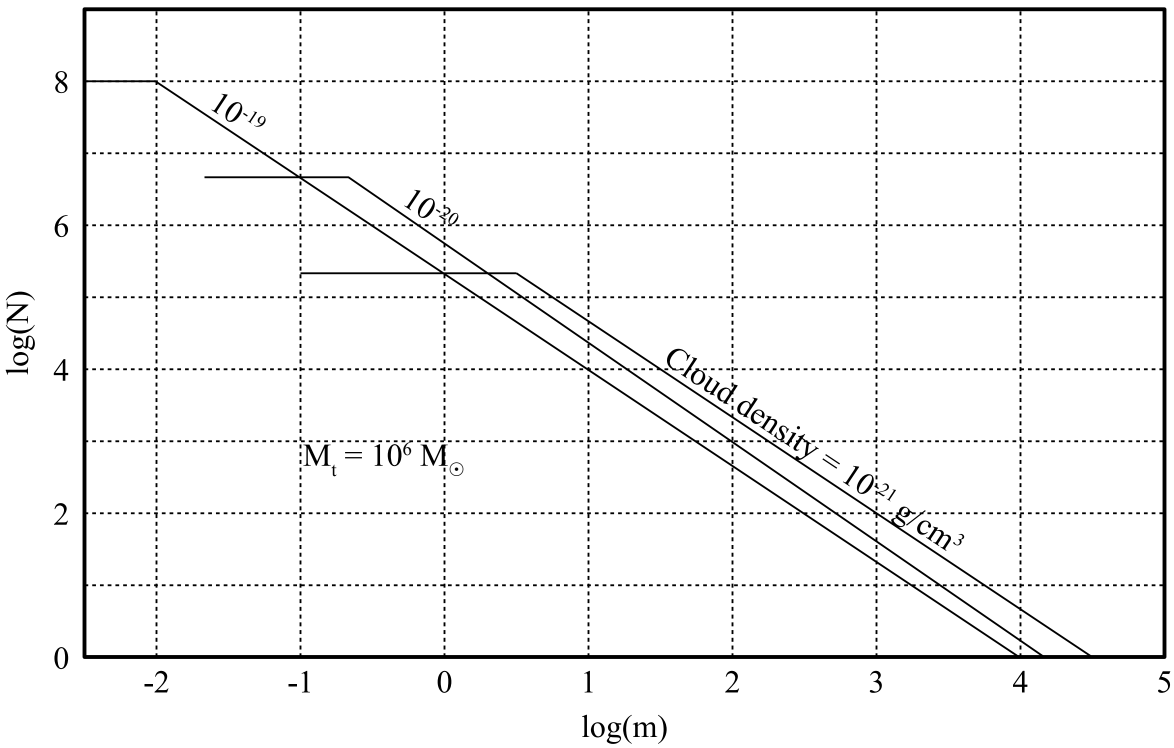

Figure 1: Total Numbers of Stars of masses greater than m, as a Function of m